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Wicked Problems and the Invention of Calculus

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Introduction

Since the 1980s, *wicked problems* have represented a category of challenges that defy clear description, cannot be addressed with existing models or theories, and resist experimentation in trying to solve them. This class of problems existed before they were identified and have been unsuccessfully addressed with Thomas Kuhn's model of scientific discovery, an expectation that requires the identification of a new object and the development of its correct interpretation. This paper proposes an alternative view of scientific discovery using the invention of Calculus as a case study that describes a successful process addressing *wicked-like* problems from a philosophical perspective, develops ideas that have an epistemological objective and are multidisciplinary in their applications, and results in additions to the Body of Knowledge that permeate human language and understanding. Leibniz's *wicked problem* was to produce a universal method of discovery at the centre of his idea of a 'General Science' and the compilation of an encyclopaedia of all knowledge available at the time. From the existing paradigm of geometrical arguments and deductive processes, there is a gestalt shift in Leibniz's *leap* in understanding mathematical methods and the language used in describing and solving problems that was rooted in the idea of infinitesimals and in a more general method of analysis. In doing so, the transition that began with his methods and notation became the first stage in a Kuhnian paradigm shift and the incorporation of Calculus and its applications into the mainstream of science. I will start by giving some background on *wicked problems* and describing the concept of discovery associated with Kuhn's ideas, and I will then introduce the process of additions to knowledge advocated in this essay. These ideas will form the antecedent to summarise the paradigm in 17th century mathematics and from there I will proceed to describe Leibniz's leap and the inherent gestalt shift that occurred in the mathematics of the 18th century. That gestalt shift was not exempt from acrimonious discussions over alternate formulations and I will present some differences between the views of Newton and Leibniz and of two of their supporters; Maclaurin and l'Hôpital. I will then describe some of the efforts that helped to expand the acceptance of Calculus and to embed it in the mainstream of science. I will conclude by proposing that there are other examples from the History and Philosophy of Science that follow a similar process of additions to the Body of Knowledge.

Wicked Problems

Wicked problems are complex, ill-structured, and resist easy resolution due to their high degree of uncertainty, ambiguity, and conflicting values. Examples include climate change, poverty, healthcare, and education reform. *Wicked problems* are typically characterised by the following:

– Complexity: They involve many interconnected elements that are difficult to disentangle.

– Uncertainty: The nature and extent of the problem is not well-defined, and there may be multiple, conflicting viewpoints on the issue.

– Ambiguity: They lack clear solutions, and the boundaries of the problem are difficult to define.

– Resistance to resolution: Because they are so complex and interconnected, there is no clear endpoint to the problem-solving process.

Although these problems were identified in the 1980s, *wicked-like* situations have long existed and been associated with Natural Philosophy. Addressing *wicked problems* requires us to reflect on the dimensions of problem-solving and confront the limits of scientific knowledge and the complexity of the systems that we seek to understand and control. It leads us to adopt a more holistic, interdisciplinary approach that integrates multiple perspectives and values, and recognizes the limits of our knowledge and the contingency of our solutions. *Wicked problems* are also multidimensional, and not just technical or scientific in nature, involving complex methodological, social, cultural, and political factors that shape our understanding of the problem and our solutions. For example, climate change involves not only science, but also complex social and political issues such as the distribution of resources, the role of technology, and the responsibility of different actors. I will now explain some ideas associated with classical views of Discovery.

Discovery and Thomas Kuhn

Until recently, it has been widely believed that any problematic situations can be approached from a professional perspective. Rittle and Webber say that "Based in modern science, each of the professions has been conceived as the medium through which the knowledge of science is applied. In effect, each profession has been seen as a subset of engineering."[1] However, Alrøe and Noe add that "...as society has become more complex, there has been a correlated differentiation of disciplines in science that results in a growing difficulty of applying science and expertise."[2] The

1 Horst W. J. Rittel, and Melvin M. Webber: "Dilemmas in a General Theory of Planning", in: *Policy Sciences,* 4 No. 2 , 1973, pp. 155 – 169.

2 Hugo Fjelsted Alrøe and Egon Noe: "The paradox of scientific expertise: A perspectivist approach to knowledge asymmetries", in: *Fachsprache – International Journal of Specialized Communication*. (3-4), 2011, pp. 152–167.

engineering approach to problem solving implies working within an existing paradigm following Thomas Kuhn's conception as a set of methods, values, and standards for success that defines a particular field and group of professionals.[3] The engineering model has traditional views of discovery, where Kuhn proposed that scientific discoveries require both a discovery-*that* and a discovery-*what*,[4] that is to say, the outcome of discovery is the identification of an entity and its correct interpretation. This approach has not been successful in addressing *wicked-like* problems and situations, and in some cases the process of discovery has followed a different path, as I describe it next.

Process of Addition to Knowledge

Discovery in the context of this paper does not contradict existing definitions and descriptions, and instead augments the understanding of its multifaceted nature, while addressing a gap in current views that fails to explain the genesis of inventions not provable or falsifiable empirically. What emerges is not a unique intellectual algorithm that produces a *thing* but rather an approach that fosters a description and understanding of a previously undefined ontological space and facilitates the development of applications, explorations, results, and solutions to problems in that space. The discovery process begins with the wish to understand a problem and generates ideas that help describe and explain said problem, even at the risk of controversy arising from challenging accepted methods and models. This creative phase becomes a catalyst in a fertile ground of diverse ideas and partial solutions in examples which adhere to existing and inadequate paradigms. Through intuition and creative integration of existing precursor ideas, a scholar invents entities (objects, models, or methods) and replicable frameworks that describe and explain them.

One key feature of *wicked problems* is their inherent value-ladenness. That is, different stakeholders hold different values, beliefs, and priorities that shape how they perceive the problem and its possible solutions. As a result, there is often disagreement about what constitutes a problem and what goals should guide its resolution. This value-ladenness makes wicked problems particularly challenging for scholars working within established paradigms, who often assume that facts and evidence can be separated from values and politics. I believe that addressing *wicked problems* requires an interdisciplinary approach that develops in stages and includes multiple stakeholders, diverse perspectives, and a willingness to embrace uncertainty and ambiguity. The result is an addition to the Body of Knowledge that permeates our vocabulary and understanding of nature and our interactions with it. I will now give an example of this point.

3 Thomas S. Kuhn, and Ian Hacking: *The Structure of Scientific Revolution*, with an introductory essay by Ian Hacking, Chicago, 2012.

4 Samuel Schindler: "Scientific Discovery: That-Whats and What-Thats", in: *Ergo,* an Open Access Journal of Philosophy 2, no. 6, 2015, https://doi.org/10.3998/ergo.12405314.0002.006

Antecedents: Historical Environment – Mathematics in the 17th Century

The antecedents to the development of Calculus rest on two areas of 17th century mathematics: geometry and infinitary arithmetic. Leibniz arrived in Paris for the first time in 1672 in a politically charged and unstable environment, and with very minimal foundations in mathematics. There he gained access to the Académie des sciences and in particular to Huygens. In addition to sharing his own work, Huygens shared with Leibniz the master work and fundamental papers from some of the best contemporary mathematical minds: Honorè Fabri, Nicolas Mercator, Blaise Pascal, James Gregory, René Descartes, and John Wallis.[5] Their efforts revolved around the works of Euclid, Apollonius, Archimedes, and Pappus; and Richard Brown argues that "...they indicated the way Mathematics should be done, what standards it should meet, the general type of problems that were of interest, and what should constitute a precise solution to them."[6] In this environment we can see echoes of the role of exemplars described in Kuhnian paradigms. Much emphasis was placed on geometric constructions of a few classical problems such as finding the area under a curve (squaring) in general, and of the circle in particular. In the late 1500s and early 1600s, some especially talented mathematicians like Françoise Viète and Christopher Clavius had begun to successfully transfer algebraic methods used previously on arithmetic problems to develop geometric constructions, however, they worked within the constraints of classical Greek mathematics which had no concept of real numbers. Numbers represented distances and dimensions, and therefore, ratios and other relations could only be expressed with homogeneous entities: segments to segments, areas to areas. The use of polynomials was limited to physical dimensions, and higher order polynomials were seen as abstractions without real applications.[7] Algebra progressed from its mediaeval use as a tool for specific problems to a symbolic language for abstraction affecting geometry applications. Multiplication of two segments was seen as an area and that of three segments as a volume. Given that multiplication was seen as a change in dimension, there was no way to represent graphically more than three dimensions.[8]

René Descartes began to transform geometry by using algebra as more than an aid in constructions by introducing equations that no longer had to be homogeneous, with the cost of adding relativity to some operations like line segment multiplication. His work, however, remained within the existing classical paradigm.[9] Descartes' publication of the Discourse on Method, Meditations on First Philosophy, and Principles of Philosophy locates mathematics and physics as antecedents to the 'order of reason,' provides their metaphysical justification and legitimisation, and

5 Siegmund Probst: "The Calculus" in: *The Oxford Handbook of Leibniz*, edited by Maria Rosa Antognazza, Oxford, 2018, pp. 213–214.

6 Richard C Brown: *The Tangled Origins of the Leibnizian Calculus; A case Study of a Mathematical Revolution*, Singapore, 2012, p. 19.

7 Idem. pp. 22–26.

8 Emily R. Grosholz: *Representation and Productive Ambiguity in Mathematics and the Sciences*. Oxford, 2007, pp 165–166.

9 Richard C Brown: *The Tangled Origins of the Leibnizian Calculus*, p. 27.

organises items of knowledge within a domain, and domains the sphere of human knowledge as a whole.[10] Descartes was not interested in the study of curves in general and did not include what we now call transcendental curves (a term introduced by Leibniz) since they were not developed from arithmetical methods and therefore were not subject to the methods he developed. Although he was committed to the Greek geometrical canons of construction and existence, his analytical geometry shows how the classical means of construction fail to provide a foundation for both geometry and the study of numbers.[11]

Leibniz was influenced by traditional proof methods and tried to follow rational paradigms by criticising Wallis for using experiments and phenomena to derive laws.[12] His work on the infinitary series was influenced by Huygens, Mercator, Wallis, Pascal, and others, and this environment forms the antecedents of Calculus. However, Leibniz did not simply produce a combination of elements existing in the fertile ground of the intellectual development of mathematics at the time. This will be my next point.

Leibniz's Leap: Infinitesimals and Generalisation

Leibniz's *wicked problem/situation* was to produce a method of discovery following Descartes: a philosophical thesis followed by examples. Leibniz published "*Maximis et Minimis…*,"[13] the seminal paper of 1684, on differential Calculus as part of his goal of developing the art of discovery at the centre of his idea of a 'General Science' and the compilation of an encyclopaedia of all knowledge available at the time.

What were some of Leibniz's root ideas? Trying to contextualise the concepts of infinity and continuum, infinitary series, infinitesimals, and indivisibility. His thinking shifted from deductive reasoning in a *continuum* to inductive, inferential thinking in a *quantised* space. This is how the linking of ideas on infinitary descriptions, conceptions, and processes connects to the need to operate with indivisibles and infinitesimals. They correspond to *ontological* conceptions that are necessary to describe and understand. The later resistance to accept Calculus was in part due to the difficulty of the transition, and in part to the uncertainty of operating with non-tangible, ill-defined *quanta*: infinitesimals/indivisibles in this case. It was necessary to ideate the *quanta* that describe the change within the process/space and the implication of using them in a framework. This did not occur in a vacuum, nor did it happen in a moment of brilliant inspiration, or as the output of an intellectual

10 Emily R. Grosholz: *Representation and Productive Ambiguity in Mathematics and the Sciences*, p. 166.

11 Idem pp. 231–232.

12 Richard C Brown: *The Tangled Origins of the Leibnizian Calculus,* Sp. 78.

13 Gottfried W. Leibniz: "Nova Methodus pro Maximis et Minimis, itemque Tangentibus, quae nec Fractas nec irrationales quantitates moratur et singulare pro illis Calculi Genus – Acta Eruditorum Octobre 1684", in: *G. W. Leibniz Naissance du calcul différentiel*, M. Parmentier (Translator). Paris, 1995, pp. 96–117.

algorithm. While Leibniz was surrounded by others in a common environment and fertile preconditions, he used pieces that existed among current discussions on ideas and integrated them in a unique way with intuition and *a priori* concepts and thoughts. He then invented a set of entities that helped define a new space and explained the implications of using them.

For Leibniz, the *leap* goes from the constraints of working with equal entities in ratios in specific cases to the description and use of infinitesimals and to the more general acceptance of numerical ratios that can then be implemented in various applications. Leibniz did not see the examples he studied with Huygens as part of his education, but rather as the point of departure for his discoveries.[14] With the publishing of the "*Maximis et Minimis…*,"[15] Leibniz leaves behind previous efforts by providing a method that works geometry differently, by applying calculation to the operations (functions in modern language) that define the curves. He was attempting to understand and mathematically describe continuous movement as seen in nature, whether in observable or unobservable entities. At issue were the adoption of different metaphysical ideas such as the corpuscular (atomic) nature of matter and the idea of continuity.

Leibniz's "*Maximis et Minimis…*,"[16] opens with a figure showing several curves and their tangents,17 proceeding to explain that the tangents can be represented by arithmetic operations on small segments ∂x, ∂w, ∂y, ∂z.[18] These small segments will come to be known as infinitesimals. Having posited the segments as represented by their numerical expressions, he proceeds to extend the operations to multiplication and division. Leibniz introduces notation that allows him to represent arbitrary segments with expressions that can be used in the same way in different curves and explains how the operations that represent the differences correspond to increases of the values associated with the curve if it is positive, or decreases if it is negative, with a maximum in a point at the apex of concavity or a minimum at a point at the bottom of a convexity.[19] Leibniz continues by calculating the differences of the differences (derivative of the derivative, or second derivative in modern language) with a notation of $\partial\,\partial x$ and uses it to associate concavity and convexity with the signs resulting from those operations. Precisely here is the departure from previous work, as the geometrical interpretation of the curve follows the numerical operations on its symbolic representation.

By analysing curves with traditional geometrical methods, approximations with geometrical figures (such as using a number of rectangles to fill the space under the

14 Marc Parmentier: "l'Optimisme Mathématique", in: *G. W. Leibniz Naissance du calcul différentiel*, M. Parmentier (Translator), pp. 12–13.

15 Gottfried W. Leibniz: "Nova Methodus pro Maximis et Minimis, itemque Tangentibus, quae nec Fractas nec irrationales quantitates moratur et singulare pro illis Calculi Genus – Acta Eruditorum Octobre 1684", in: *G. W. Leibniz Naissance du calcul différentiel*, M. Parmentier (Translator). Paris, 1995, pp. 96–117.

16 Ibidem.

17 Ibidem.

18 Idem pp. 105–106.

19 Idem p. 107.

curve), the introduction of errors was inevitable. Numbers were necessarily different unique entities that needed to be connected to a continuous line. Leibniz approached the problem by looking at quantities as representing segment lengths that were very, very small, and yet different from zero. Using these "vanishing," intangible, incommensurable entities born from the mathematical description of the curves themselves, Leibniz could approximate the actual line so closely that the error disappeared for all practical purposes. Separating the infinitesimals from the physical and geometrical interpretation of the curve was a radical departure from scientists requiring experimentation and data to prove hypotheses, and from mathematicians providing rigorous proofs using Euclidean geometry, thus creating an intellectual space where it was necessary to address and use entities that were not representational of reality, and with no empirical status.[20]

Here is the departure from previous work, as the geometrical interpretation of the curve follows the numerical operations on its symbolic representation. The behaviour of the curve can be described by numerical analysis, and in doing so, Leibniz is creating an epistemic entity informed by mathematics in the study of the curve. The description of the method is semantic, teleological, and pedagogical. Leibniz develops not only a notation but a vocabulary that describes the behaviour of the curve, defining maximums and minimums by the arithmetical result that identifies them and by the use of descriptors such as inflection point (punctum flexus contrarii)[21]. He anticipates the use of the method in a variety of applications by developing rules and examples in the calculations of examples from power and rational functions, and radical functions. Leibniz characterises his method as algorithmic (algorithmo)[22] and calls it 'differential' (differentiali).[23] He elaborates on the usefulness of the approach by arguing that it can be extended to all curves without relying on hypotheses specific to particular curves or families of curves. In doing so, Leibniz goes beyond the specificity of a solution of a problem to the universality of a method that can be explicitly used in a myriad of applications and fields. In his defence of the method, Leibniz explains that in progressively difficult problems, calculations by traditional methods might considerably increase in complexity, which his approach avoids, while using a notation that is not particular to a specific case.[24] The approach above also relaxes the restrictions when operating within specific problems such as movement, which would ease the realism requirement to understand differentials in terms of variations in time such as was used by Newton, and in doing so it permits a true generalised analysis for the first time. I

20 Mitchell G. Reyes: "The rhetoric in mathematics: Newton, Leibniz, the Calculus, and the rhetorical force of the infinitesimal", in: *Quarterly Journal of Speech*, 90:2, 2004, 163–188, https://doi.org/10.1080/0033563042000227427 pp. 178–179.

21 Gottfried W. Leibniz: "Nova Methodus pro Maximis et Minimis, itemque Tangentibus, quae nec Fractas nec irrationales quantitates moratur et singulare pro illis Calculi Genus" Acta Eruditorum Octobris, 1684, pp. 467–473. p. 468. Retrieved from Mathematical Association of America Mathematical Treasure: *Leibniz's Papers on Calculus – Differential Calculus*.

22 Idem, p. 469.

23 Ibidem.

24 Idem. pp. 471–472.

will now describe how these ideas represented the beginning of a mathematical revolution à la Kuhn.

Gestalt Shift and the New Mathematics

There is a gestalt shift in Leibniz's *leap* in understanding mathematical methods and the language used in describing and solving problems. Severing the attachment to *reality* allowed an understanding that was disconnected from strictly geometric arguments and deductive logic to infer the applicability of the method beyond specific cases. The difficulty in transitioning from infinitary techniques in series to algebraic methods in integrals of 'functions' (in modern terms) arose from the reluctance to accept their equivalence, and thus, accepting a break with existing paradigms in 17^{th} century mathematics. The transition uses the concept of infinitesimals as a bridge, and it was the philosophical approach that made possible the break with the existing paradigm by developing a general problem-solving 'process' and a general 'language' that permitted seamless communication and diffusion. The ease of implementation of the new methods and their success in solving existing and new problems and applications is one of the critical factors in their eventual adoption.

By 1676, Leibniz had long recognised that the summation of the terms of a series is inverse to their difference, and yet, there were no systematic applications to quadratures and tangents.[25] The link between these concepts is the essence of the usefulness and novelty of Calculus. What Leibniz put together was a notation and algorithm to solve a classical problem in a manner that could become a *universal method* and could be applied to many problems beyond specific cases and the set of classical problems. In modern integral Calculus we teach that a Riemann sum (in the limit with an infinite number of terms) is the same as the integration of a function that describes the curve. This is an example of the power of Calculus:

$$if\ (b-a)/n = \Delta x,\ then\ \lim_{n\to\infty} \sum_{i=1}^{n} f(xi)\Delta x = \int_{a}^{b} f(x)\delta x \quad (1)$$

The notation on the left side of equation 1 describes an infinitary sum whose origin is the quadrature (area under the curve) of a specific curve. The notation on the right side describes the area under the curve, found by adding infinite rectangles that approximate the full space under the curve (giving origin to the infinitary series on Dthe left). The sigma notation (left) is a tool to write more efficiently the description of the solution of the problem, and is solved with a formula as a method to efficiently carry out the summation of the series, such as the formula to add consecutive numbers in an arithmetic series. The integration formula in a definite integral (right) allows for an algorithmic solution of a family of functions that makes it easier to obtain the area under the curve (the quadrature) without recurring to infinite series

25 Richard C Brown: *The Tangled Origins of the Leibnizian Calculus*, p. 126.

techniques. If we have two different descriptions of a solution of a problem that produce indistinguishable solutions, then it does not matter which one we choose since they are both true representations of reality. The system is so powerful that to solve the problem it is only necessary to apply the algorithm, without necessarily understanding the details of the problem or being an advanced mathematician.

The gestalt shift originated by Leibniz's invention and subsequent adoption, expanded with the work, pedagogical methods, and communication from key figures like l'Hôpital, the Bernoullis in the late 17th century, and later with Euler's text and work in the 18th century. The adoption of Calculus stemmed from a Kuhnian revolution in mathematics, with fierce challenges and a lengthy process of adoption in which political, sociological, historical, and pedagogical factors played a role. I will next describe some of the differences in the alternate views of Newton and Leibniz.

Challenge from Newton and how their Calculus Differ

Calculus faced bitter challenges, particularly from Newton and his followers. Reyes argues that the adoption of the Calculus developed by Leibniz and Newton evolved from a network of discourses and brought with it a change in how humans perceive, think about, and discuss nature, and it became accepted knowledge through wide communication, discussion, and clarification between established thinkers.[26] From a philosophical perspective, the differences between Leibniz's and Newton's approaches to Calculus reflect their broader philosophical views. Leibniz saw infinitesimals as a tool for reasoning about the behaviour of functions and believed that they could be used to develop a rigorous foundation for Calculus. In contrast, Newton was an empiricist who believed that knowledge should be based on observation and experimentation. Leibniz believed in a relational theory of space and time, in which they are defined by the relationships between objects. He saw infinitesimals as a way to reason about these relationships and believed that they could be used to develop a unified theory of physics. Newton, on the other hand, believed space and time were absolute, and existed independently of objects. He saw his Calculus as a tool for understanding the behaviour of physical systems in this absolute framework. Newton's views and conceptions as a physicist were constrained by a mathematical language specific to the problem at hand while Leibniz's approach was philosophical and concerned with developing explanations in a universal language not restricted to specific problems. This is where the leap to a more general method and understanding breaks with the existing paradigm, opening the path for its use in other applications.

26 Mitchell G. Reyes: "The rhetoric in mathematics: Newton, Leibniz, the Calculus, and the rhetorical force of the infinitesimal", in: *Quarterly Journal of Speech*, 90:2, 2004, 163–188, https://doi.org/10.1080/0033563042000227427, p. 165.

Bos explains the nature of the difference between Leibniz's and Newton's approaches to Calculus.[27] This is why we think of Leibniz as a mathematician and Newton as a scientist, no matter how brilliant a mathematician he was. Newton's mathematical methods were extremely difficult to replicate and generalise, and therefore were not adopted, while Leibniz's were used extremely successfully by others and over time became the new methodological standard, the new paradigm. The approach could be applied to higher order differentials and sums, and while it was a logical next step in the application of the method at first it was difficult to see phenomena explained by the mathematical operations. Instead, explanations began with mathematical descriptions that could later explain and predict phenomenological behaviour. This is crucial to understand the gestalt shift in mathematics in the 18th century. Replicability and mechanisation of approach made it possible for the new mathematics to be generally used. What made it difficult for Newton's followers to replicate was a combination of the difficulty in the language used to describe his method and the specificity of its goal. Despite the later efforts to generalise its results and methods by Colin Maclaurin (1698–1746)[28] and others, it was not possible for British mathematicians to emulate or expand its use and this stagnated British mathematics for more than a century. I agree with Richard Brown when he states that more than the physical discoveries of Newton, it was Leibniz's Calculus that made the modern world possible.[29] Next I will present some examples of the pedagogical approaches used in explaining the alternate formulations of Calculus.

Expansion: Pedagogical differences – Maclaurin vs l'Hôpital

The change in paradigm did not happen without bitter controversies and discussions in a revolutionary process à la Khun. In the debate about Calculus and its implementation and uses, the different approaches of Newton and Leibniz had implications for the influence and acceptance of the new tools in science and mathematics. The second phase of the epistemological process of additions to the Body of Knowledge implies the use of accepted semantic entities. Similarly to Niels Bohr's description of the *Principle of Correspondence*, we rely on classical views rooted in our experience in order to accept the use of new entities and frameworks. The semantic articulation of these concepts is achieved through socialisation of the ideas, and it triggers discussion and comparisons with alternate views. The usefulness of the semantics and notation used to describe and apply to new situations becomes critical in their acceptance and diffusion. Good examples of the exchanges that took place during this phase are some of the arguments from Maclaurin regard-

27 Henk J. M. Bos: "Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus", in: *Archive for History of Exact Sciences* 14, no. 1 (1974): 1–90. http://www.jstor.org/stable/41133417.

28 Colin Maclaurin: *An account of Sir Isaac Newton's Philosophical Discoveries: in four books*, London, 1747.

29 Richard C Brown: *The Tangled Origins of the Leibnizian Calculus*.

ing Newton's methods and ideas and the success in acceptance of Leibniz's Calculus vs Newton's: Maclaurin defends Newton's slowness to publish as his reluctance to engage in controversy out of modesty.[30] Maclaurin contrasts Newton's explanations of gravity with Leibniz's, where the former seemed to lean towards physics and the latter towards mathematics and philosophy. Maclaurin explains Leibniz's success by virtue of his theological argument that God (*the Deity*) had made a most perfect world, and that this argument resonated with others.[31] Maclaurin explains Leibniz's introduction of *monads*[32] and afterwards proceeds to define space, body, etc., in an approach to the *continuum vs quanta* issue.[33] Following the reasoning of science vs mathematics/philosophy in the two approaches, he leads the discussion and defence of Newton's views by explaining their root in the study of motion.[34] In a separate text explaining Newton's theory of fluxions, he describes indivisibles,[35] followed by a description of the *continuum*, metaphysical arguments, and the idea of infinitesimals.[36] Maclaurin wrote the *Treatise of Fluxions* to answer Berkeley's attack on Newton's methods for their lack of rigour.[37] These arguments are either apologetical or seek to clarify Newton's language, and contrast with expositions in the texts from l'Hôpital and others that concentrate on didactic descriptions of the implementation of Leibniz's methods in a variety of problems.

The first Calculus textbook was written by l'Hôpital[38] and follows the pedagogical approach and structure of modern texts, with a presentation of several problems and their solutions. For example, section IX, proposition 1 presents the problem of a fraction where both numerator and denominator approach zero and where the ratio is the same as that of the differentials at the same point. The description of the solution that follows the stated problem is known as the l'Hôpital Rule (also attributed to Bernoulli) and is an example of the tone and purpose of the text which contributed to the expansion of Calculus. l'Hôpital's approach was successfully used in communications and publications of other scholars addressing different applications, thus contributing to the acceptance of Leibniz's Calculus notation and methods which I will present next.

30 Colin Maclaurin: *An account of Sir Isaac Newton's Philosophical Discoveries: in four books*, London, 1747, pp. 10–15.

31 Idem, p. 80.

32 Idem, p. 81.

33 Idem, p. 100.

34 Idem, p. 104.

35 Colin Maclaurin: *A Treatise of Fluxions in Two Books*, Edinburgh, 1742, p. 1

36 Idem, pp. 39–45.

37 J J O' Connor and E F Robertson: "Colin Maclaurin", *Maths History at Saint Andrews*. https://mathshistory.st-andrews.ac.uk/Biographies/Maclaurin/

38 Guillaume François Antoine de l'Hôpital: *Analyse des infiniment petits, pour l'intelligence des lignes courbes.* A Paris, 1716.

Expansion: Communications and Applicability; Bernoullis, Varignon, Bodenhausen

The process of addition to knowledge stems from active and confrontational debates on the articulation of the new ideas. Understanding emerges from a dialectical approach within the scientific community before new knowledge gains the status of a correct conceptualization. Concepts, methods, and their interpretation then spread through textbooks and teaching and by their use in new applications. An example of this phase is seen in the arguments on the validity of the concept of infinitesimals, and the discussion of usefulness of the new methods as in the exchanges between Leibniz and Huygens.

The successful adoption of Calculus arose from the communications with others in the development of applications such as differential equations of the type $\partial y=(2x+2y)\,\partial x$ (communications with Jacob Bernoulli), the integration of polynomial and root functions (with Johann Bernoulli), and the determination of central forces from orbits (with Varignon). Further diffusion and acceptance ensued from the exchange of ideas, for example between Leibniz and Rudolph von Bodenhausen, regarding the study of the catenary and an approximation to the value of *e*,[39] and preparations for a pedagogical text in 1690.[40] There are additional sociological factors influencing the incorporation of Calculus to the mainstream of Science and the Body of Knowledge, as I will briefly present next.

Expansion: Adoption; Euler and the French Schools

Wahl has explored the role of cosmopolitanism, nationalism, and expatriates in the adoption and expansion of Calculus.[41] Textbooks like those of l'Hôpital[42] and later those of Euler,[43] L. A. de Bougainville, and Maria Gaetana Agnesi[44] succeeded in spreading its use among different audiences. During the 18th century, problems in celestial mechanics, hydrodynamics, elasticity, and in general rational mechanics grew in importance and scope with interest expanding and expeditions sent to check

39 Michael Rough and Siegmund Probst: "The Leibniz catenary and approximation of e – an analysis of his unpublished calculations", in: *Historia Mathematica* 49, 2019, 1–19 https://doi.org/10.1016/j.hm.2019.06.001

40 Rudolph von Bodenhausen, in: *Leibniz Manuscripts on Mathematics LH35*, 11, 18c, 1690 http://digitale-sammlungen.gwlb.de/resolve?id=00068193.

41 Charlotte Wahl: "Between cosmopolitanism and nationalism. The role of expatriates in the dissemination of Leibniz's differential Calculus", in: *Almagest,* 2014, 40–68. https://doi.org/10.1484/J.ALMAGEST.5.103566.

42 Guillaume François Antoine de l'Hôpital: *Analyse des infiniment petits, pour l'intelligence des lignes courbes*. A Paris, 1716.

43 Leonhard Euler: *Introduction a l'analyse infinitésimale,* Traduite du latin en français, avec des notes & des éclaircissements – Tome Premier – par J. B. Labey. A Paris, 1796.

44 Henk J. M. Bos: "Calculus in the eighteenth century – the role of applications" in: *Lectures in the History of Mathematics – History of Mathematics Volume 7 – Chapter 7.* American Mathematical Society 1995. DOI: https://doi.org/10.1090/hmath/007, pp. 113–128.

results. Rational mechanics provided a language and the concepts for the new methods of analysis, and with these influences, it conferred prestige on the new methods of Calculus.[45] In France, influential military and civil engineering educational institutions adopted textbooks on hydraulics. By the end of the century, the knowledge of pure mathematics required for admission increased along with what was taught in those schools, and the influential École Polytechnique made Calculus part of the curriculum,[46] thus establishing it and analysis within the mainstream of academia and the professions. I will now conclude this essay by summarising how the example follows the proposed view of discovery as well as suggesting other examples that exhibit similar patterns.

Conclusion

Understanding the invention of Calculus goes beyond the how and must address the important question of the why. Newton was not interested in 'philosophy' and instead focused on the description of *how* and in the specific problem of motion, and therefore his solution was particular to that problem. Leibniz was driven by the *why*, by philosophical questions, and by the need to contribute to the improvement of society, and he was working on the *wicked problem* of developing a universal method for discovery and a language to communicate it. In teaching Calculus, I show how we transition from the methods for solving specific problems (limits in the calculation of slopes, and infinite series in the calculation of areas) to the efficient description and general methods of solutions contained in derivatives and integrals as a framework for working on new problems in the various fields associated with their studies. Derivatives allow us to understand how quickly something changes, and integrals let us find out about the extent of that change. The goal is to promote the transition from the practitioner approach of a science student who replicates existing and known solutions to that of a scientist who uses frameworks, methods, and diverse entities in addressing new situations, problems and applications in science. Replicability, implementability, and pedagogical approach became the roots of the adoption of Calculus and analysis as a new paradigm beginning in the 18th century, and later, of language and understanding in the mainstream in Science.

Leibniz's ideas can be translated and seen in examples far removed from technical solutions in mathematics; the idea of speed and amount of change being related to each other, integration as a seamless sum of parts, the relation between continuum and quanta in understanding a process, the importance of normative language in understanding, the applicability of visceral/ physical images in describing the process and understanding its implications and role in solutions, the philosophical implications of choosing seemingly different explanations and solution meth-

45 Idem, p. 119.
46 Idem, pp. 121–122.

ods that nonetheless produce indistinguishable results. These are all present in Calculus, and they exist also in Charles Darwin's ideas on evolution, complementarity from Niels Bohr, computability and mechanical intelligence from Alan Turing, and in information theory from Claude Shannon. The resistance to Calculus was due in part to the difficulty of the transition, in part to the uncertainty of operating with non-tangible, ill-defined *quanta*: infinitesimals/ indivisibles in this case, but applicable to other cases following this process: quanta, infinitesimals, bits of information, evolutionary traits and steps, Turing machine (computer software) steps, and first order logic rules in the computability of numbers. For additions to the Body of Knowledge following the path described in this essay, it is necessary to ideate the *quanta* that describe the change within the process/space and the implication of using it in a framework. This does not occur in a vacuum, nor does it happen in a moment of brilliant inspiration, or as the output of an intellectual algorithm. This is what makes them additions to the Body of Knowledge in a different way and of a different type from the *that* and *what* of traditional conceptions of discovery. Those concepts are not extensions or syntheses of precursors of the scholar's work, but true inventions of new ontological spaces that expanded our understanding.